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# ON THE MAXIMAL SMOOTHING EFFECT FOR MULTIDIMENSIONAL SCALAR CONSERVATION LAWS

PIERRE CASTELLI & STÉPHANE JUNCA

ABSTRACT. In 1994, Lions, Perthame and Tadmor conjectured an optimal smoothing effect for entropy solutions of multidimensional scalar conservation laws. This effect estimated in fractional Sobolev spaces is linked to the flux nonlinearity. In order to show that the conjectured smoothing effect cannot be exceeded, we use a new definition of a nonlinear smooth flux which proves efficient to build bespoke explicit solutions. First, one-dimensional solutions are studied in fractional  $BV$  spaces which turn out to be optimal to encompass the smoothing effect: regularity and traces. Second, the multidimensional case is handled with a monophasic solution and the construction is optimal since there is only one choice for the phase to reach the lowest expected regularity.

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## 1. INTRODUCTION

For the multidimensional scalar conservation laws

$$(1.1) \quad \partial_t U + \operatorname{div}_X F(U) = 0, \quad U(0, X) = U_0(X) \in L^\infty(\mathbb{R}^n, \mathbb{R})$$

the first smoothing effect measured in Sobolev spaces was obtained in 1994 by Lions, Perthame and Tadmor ([19]) for a flux  $F \in C^1(\mathbb{R}, \mathbb{R}^n)$ . It was improved by Tadmor and Tao in 2007 ([24]). This smoothing effect generalizes the  $BV$  smoothing effect obtained in 1957 independently by Lax and Oleinik for a one-dimensional uniformly convex flux

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([18, 21]). In [19] the regularity is measured in the Sobolev space  $W_{loc}^{s,1}(\mathbb{R}_X^n, \mathbb{R})$  with a small  $s \in ]0, 1[$ : Lions, Perthame and Tadmor conjectured that

$$s = \alpha$$

where  $\alpha \in ]0, 1[$  (Definition 2) quantifies the nonlinearity of the flux on the compact interval  $K = [\inf U_0, \sup U_0]$ . In the one-dimensional case, De Lellis and Westdickenberg showed in 2003 that  $s \leq \alpha$  for power-law convex fluxes ([11]) and Jabin showed in 2010 that  $s = \alpha$  for  $C^2$  fluxes under a generalized Oleinik condition ([13]).

For a nonlinear multidimensional smooth flux the parameter  $\alpha$  is determined explicitly in [16] with an equivalent definition of nonlinearity recalled in Section 2 below. In particular the parameter  $\alpha$  depends on the space dimension  $n$  and satisfies:  $\alpha \leq \frac{1}{n}$ . Moreover, Definition 4 naturally yields the construction of a supercritical family of oscillating smooth solutions -on a bounded time before shocks- exactly uniformly bounded in the optimal Sobolev space conjectured ([16]).

In this paper:

- we obtain an extension of the inequality  $s \leq \alpha$  for all nonlinear multidimensional smooth fluxes;
- we present examples of special individual solutions (and not a family of solutions as in [16]) which belong to the almost optimal Sobolev space.

In order to do so we use the fractional  $BV$  spaces which appear to be more relevant in the one-dimensional case to get the regularity and the shock structure of entropy solutions ([1]). One-dimensional examples with low regularity given in [2, 6, 11] are first studied in generalized  $BV$  spaces and then extended to the multidimensional case. Notice that the construction is optimal for the one-dimensional case, at least for the class of degenerate strictly convex fluxes ([1, 2, 11]). As in [2, 5, 8] these examples are not related to the convexity. We conjecture that it is also optimal for the multidimensional case, at least for fluxes smooth enough (of class  $C^{n+1}$ ). For a flux only of class  $C^1$  the natural way is to generalize the  $BV_\Phi$  approach developed in [3].

The main result of the paper is about the limitation of the smoothing effect.

**Theorem 1. [Solutions with the minimal Sobolev regularity expected]**

*Let  $K \subset \mathbb{R}$  be a proper compact interval,  $F \in C^\infty(K, \mathbb{R}^n)$  a nonlinear flux such that the associated  $\alpha = \alpha[K]$  is positive. Then, for all  $\varepsilon > 0$  and for all  $T > 0$ , there exists an entropy solution  $U$  with values in  $K$  such that for all  $t \in ]0, T[$ ,*

$$U(t, \cdot) \in W_{loc}^{\alpha-\varepsilon,1}(\mathbb{R}^n, \mathbb{R}) \quad \text{but} \quad U(t, \cdot) \notin W_{loc}^{\alpha+\varepsilon,1}(\mathbb{R}^n, \mathbb{R}).$$

To prove this main result the paper is organized as follows. Two definitions of a nonlinear flux are recalled in Section 2. Fractional and generalized  $BV$  spaces,  $BV^s$  and  $BV_\Phi$ , are introduced in Section 3. We make comparisons with the fractional Sobolev spaces and we give in Proposition 10 an explicit way to compute the generalized total variation in some particular cases. Section 4 deals with optimal examples with low regularity in  $BV^s$ . The multidimensional case is handled in Section 5 to conclude the proof of Theorem 1.

## 2. NONLINEAR FLUX

There have been several definitions of a nonlinear flux depending on the regularity of the flux: [12, 18, 21] for a  $C^2$  flux, [19] for a  $C^1$  flux, [22, 25] for a  $C^0$  flux. These definitions are compared in [16]. For an analytic flux they are equivalent with recent Definition 4 below. The first definition related with the smoothing effect for multidimensional conservation laws was given by Lions, Perthame and Tadmor:

**Definition 2. [Nonlinear flux [19]]** Let  $M$  be a positive constant.  $F : \mathbb{R} \rightarrow \mathbb{R}^n$  is said to be nonlinear on  $[-M, M]$  if there exist  $\alpha > 0$  and  $C = C_\alpha > 0$  such that for all  $\delta > 0$

$$\sup_{\tau^2 + |\xi|^2 = 1} |W_\delta(\tau, \xi)| \leq C \delta^\alpha,$$

where  $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$  and  $|W_\delta(\tau, \xi)|$  is the one-dimensional measure of the singular set:

$$|W_\delta(\tau, \xi)| := \{ |v| \leq M, |\tau + a(v) \cdot \xi| \leq \delta \} \subset [-M, M] \quad \text{and} \quad a = F'.$$

In all the sequel, only the greatest  $\alpha$  is considered.

**Example 3.** If  $f$  is a scalar power-law flux:  $f(u) = \frac{|u|^{1+d}}{1+d}$  for  $d \geq 1$ , then the greatest  $\alpha$  on  $[-M, M]$  is  $\frac{1}{d}$ . Burgers' flux corresponds to  $d = 1$ .

The construction of solutions with minimal regularity uses a precise understanding of the nonlinearity. A new definition appears for the first time in [4] for a genuinely nonlinear vectorial flux and then in [7, 9, 13, 16].

**Definition 4. [Nonlinear smooth flux [16]]**  $F \in C^\infty(\mathbb{R}, \mathbb{R}^n)$  is nonlinear on  $K$  if for all  $U \in K$  the quantity

$$d_F[U] = \min \{ j \in \mathbb{N}^* \mid \text{rank}(F''(U), \dots, F^{(1+j)}(U)) = n \}$$

is finite. Moreover,  $d_F[\cdot]$  admits a maximum on  $K$  for some  $\underline{U} \in K$  which quantifies the nonlinearity by the integer:

$$d_F = \sup_{u \in K} \min \{ j \in \mathbb{N}^* \mid \text{rank}(F''(U), \dots, F^{(1+j)}(U)) = d \} = d_F[\underline{U}].$$

The flux  $F$  is genuinely nonlinear on  $K$  if  $d_F = n$ .

Following [16] this condition also means that the curve  $\Gamma = \{A(V), V \in K\}$  never stays in any hyperplane. In some sense  $d_F$  measures the degeneracy of the flux. Notice that by definition  $d_F[U] \geq n$  and the genuine nonlinearity means that for all  $U \in K$ ,  $d_F[U] = n$ , so that the family  $\{F''(U), \dots, F^{(1+n)}(U)\}$  is a basis of  $\mathbb{R}^n$ . The constant state  $\underline{U}$  will play a key role later: the most singular entropy solutions built are near  $\underline{U}$ .

*Remark 5.* In dimension 1 this definition reduces for the scalar flux  $f$  to the first non zero derivatives of  $a(u) = f'(u)$ :

$$d_F = \sup_{u \in K} \min \{ j \in \mathbb{N}^* \mid a^{(j)}(u) \neq 0 \}.$$

**Theorem 6.** [[16]] *If  $F \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ , then  $\alpha = \frac{1}{d_F} \leq \frac{1}{n}$ .*

Moreover, the parameter  $\alpha$  is the inverse of an integer greater than the space dimension. The particular case  $\alpha = \frac{1}{n}$  corresponds to the maximal nonlinearity, namely the genuine nonlinearity [4, 7, 9, 13].

### 3. $BV^s$ AND $BV_\Phi$ SPACES

What is the right functions space to measure properly the regularity of entropy solutions? This natural question is asked by Tartar in [26]. Sobolev spaces are considered not to be the optimal ones in [13]. Sobolev spaces do not provide the fundamental traces property of  $BV$  functions ([7, 9, 27]). In the one-dimensional case,  $BV^s$  spaces provide a relevant framework ([1, 2]): the right fractional exponent of entropy solutions is reached and the like- $BV$  structure is recovered. Some other regularities are derived from the  $BV^s$  regularity, for instance the  $BV$  regularity of  $\varphi(u)$  in [8, 23] where  $u$  is an entropy solution and  $\varphi$  a nonlinear function. Moreover, the  $BV^s$  regularity yields a  $W^{s-\varepsilon,1}$  regularity for all  $\varepsilon > 0$ . Thanks to Proposition 10 below we get a simple and sharp exponent  $s$  for the examples presented in Section 4. These examples are a little more regular than expected. The surplus of regularity can be reduced as much as desired. An example adapted from [11] seems optimal in the  $BV^s$  framework. By using a finer measurement of the regularity with the larger class of  $BV_\Phi$  spaces including the  $BV^s$  spaces, it is shown that the last example is still a little more regular than the critical regularity. It is the reason why we also use the  $BV_\Phi$  spaces. The definitions of these generalized  $BV$  spaces are briefly recalled. The reader is referred to [20] for the first extensive study of the  $BV_\Phi$  spaces.

**Definition 7.** [ $BV_\Phi$  spaces [20]] Let  $I$  be a non-empty interval of  $\mathbb{R}$  and let  $\mathcal{S}(I)$  be the set of subdivisions of  $I$ :  $\{(x_0, x_1, \dots, x_n), n \geq 1, x_i \in I, x_0 < x_1 < \dots < x_n\}$ . Let  $M > 0$ ,  $\Phi$  an even convex function on  $[-2M, 2M]$ , positive on  $]0, 2M]$  such that  $\Phi(0) = 0$  and  $u$  a function defined on  $I$  such that  $|u| \leq M$ .

i) The  $\Phi$ -variation of  $u$  with respect to the subdivision  $\sigma = (x_0, x_1, \dots, x_n)$  is:

$$TV_\Phi u[\sigma] = \sum_{i=1}^n \Phi(u(x_i) - u(x_{i-1})).$$

ii) The total  $\Phi$ -variation of  $u$  on  $I$  is:

$$TV_\Phi u[I] = \sup_{\sigma \in \mathcal{S}(I)} TV_\Phi u[\sigma].$$

iii) If  $\Phi$  satisfies the condition

$$(\Delta_2) \quad \exists h_0 > 0, k > 0, \Phi(2h) \leq k\Phi(h) \quad \text{for } 0 \leq h \leq h_0,$$

then  $BV_\Phi(I) := \{u : I \rightarrow \mathbb{R}, |u| \leq M, TV_\Phi u[I] < \infty\}$  is a linear space.

Else we set  $BV_\Phi(I) := \{u : I \rightarrow \mathbb{R}, \exists \lambda > 0, TV_\Phi(\lambda u)[I] < \infty\}$ , which is a metric space.

*Remark 8.* 1) According to the assumptions made on  $\Phi$ , it is necessarily an increasing function on  $]0, +\infty[$ .

2) Notice that [20] considers the case  $\Phi(u) = o(u)$  near 0, which leads to a less regular space than  $BV$ :  $BV \not\subset BV_\Phi$ . The case where  $\Phi(u) = u$  or  $\Phi(u) \sim u$  near 0 yields  $BV = BV_\Phi$ . For degenerate fluxes we are in the context of [20]:  $\Phi(u) = o(u)$  near 0.

3) In the particular case where  $\Phi$  is a power function:  $\Phi(u) = |u|^{1/s}$  with  $s > 1$ , then  $BV_\Phi(I) = BV_{|u|^{1/s}}(I)$  is  $BV^s(I)$  and for  $s = 1$ ,  $BV^s(I) = BV(I)$ , the space of functions of bounded variation.

**Example 9.** 1) Let  $\Phi(u) = \exp\left(-\frac{1}{u^2}\right)$ ,  $|u| \leq 1$ . Since  $\Phi(u) = o(|u|^\alpha)$  for all  $\alpha \geq 1$ , it follows that for all  $s \in ]0, 1]$ ,  $BV^s \subset BV_\Phi$ . In particular,  $BV^s \neq BV_\Phi$  for all  $s \in ]0, 1]$ .  
 2) Let  $\Phi(u) = -\frac{|u|^\alpha}{\ln|u|}$ ,  $|u| < 1$ ,  $\alpha \geq 1$ ,  $s = \frac{1}{\alpha}$ . The following inclusions hold for all  $\varepsilon > 0$ :  $BV^s \subset BV_\Phi \subset BV^{s-\varepsilon}$ .

Proposition 2.3 p.660 in [1] is generalized here in the  $BV_\Phi$  framework. The  $BV^s$  optimal norm is less easy to get than in  $BV$ . Fortunately, for a function which is alternatively increasing and decreasing with less and less oscillation, the total  $\Phi$ -variation is estimated as in  $BV$ .

**Proposition 10. [ $TV_\Phi$  for oscillation with decreasing amplitudes]**

Let  $(x_k)_k$  an increasing sequence,  $I_k = [x_k, x_{k+1}]$ ,  $I = \cup_k I_k$ . If  $u$  is a monotonic function on all  $I_k$  such that the algebraic amplitude on  $I_k$ :  $\delta_k = u(x_{k+1}) - u(x_k)$  satisfies  $\delta_{k+1}\delta_k \leq 0$  and  $|\delta_{k+1}| \leq |\delta_k|$ , then  $TV_\Phi u[I] = \sum_k \Phi(\delta_k)$ .

*Remark 11.*

1) These points  $x_k$  will be called *extremal points* subsequently.

2) Notice that the two conditions  $\delta_{k+1}\delta_k \leq 0$  and  $|\delta_{k+1}| \leq |\delta_k|$  are compulsory to get the total  $\Phi$ -variation. Else the strict inequality  $TV_\Phi u[I] > \sum_k \Phi(\delta_k)$  occurs, as shown

by the two counterexamples below. These conditions are related to the strict convexity of  $\Phi$  and the fact that  $\Phi(0) = 0$ , which yields in particular the following inequality:

$$(3.1) \quad \Phi(a) + \Phi(b) < \Phi(a+b) \text{ when } a > 0, b > 0.$$

i) If  $\delta_{k+1}\delta_k > 0$ , then a strictly monotonic function provides a counterexample. Set  $u(x) = x$ ,  $x_0 = -1$ ,  $x_1 = 0$ ,  $x_2 = 1$ . Then  $\sum_k \Phi(\delta_k) = 2\Phi(1) < \Phi(2) = TV_\Phi u[-1, 1]$ .

ii) If  $(|\delta_k|)_k$  is not decreasing, then consider  $u$  a continuous piecewise linear function such that  $|u'(x)| = 1$  on  $]x_k, x_{k+1}[$ ,  $x_0 = -1$ ,  $x_1 = 0$ ,  $x_2 = \varepsilon$ ,  $x_3 = 1 + \varepsilon$ . So up to a constant:  $u(x) = x$  on  $I_0$ ,  $u(x) = -x$  on  $I_1$  and  $u(x) = x - 2\varepsilon$  on  $I_2$ . Then  $\sum_k \Phi(\delta_k) = 2\Phi(1) + \Phi(\varepsilon) < \Phi(2) = TV_\Phi u[-1, 1]$  for  $\varepsilon$  small enough.

*Proof.* Set  $y_1 < \dots < y_r$  a subdivision of  $I$ . We have to prove that:

$$\sum_{i=1}^{r-1} \Phi(u(y_{i+1}) - u(y_i)) \leq \sum_k \Phi(\delta_k).$$

The strategy consists of modifying the initial subdivision in order to increase its  $\Phi$ -variation at each step of the construction. We will successively:

- i) reduce the subdivision by removing some intermediary points ;
  - ii) replace some points of  $(y_i)_{1 \leq i \leq r}$  by extremal points and add if necessary some consecutive extremal points.
- i) Assume that three points  $y_i, y_{i+1}, y_{i+2}$  are in the same interval  $I_k$ . It follows from the monotonicity of  $u$  on  $I_k$  and the strict convexity of  $\Phi$  that:

$$\Phi(u(y_{i+1}) - u(y_i)) + \Phi(u(y_{i+2}) - u(y_{i+1})) \leq \Phi(u(y_{i+2}) - u(y_i)),$$

so the intermediary point  $y_{i+1}$  has to be removed from the initial subdivision to obtain a larger  $\Phi$ -variation. Repeating this reduction as many times as necessary, there are finally at most two points of the new subdivision in each interval  $I_k$ .

ii) The second step of the construction focuses on the oscillations of the function  $u$ . It follows from the decreasing-amplitude assumption that the sequences  $(u(x_{2k}))_k$  and  $(u(x_{2k+1}))_k$  are monotonic and correspond to the local extrema of the function  $u$ . To set the monotonicity, assume for instance  $\delta_0 < 0 < \delta_1$  (else replace  $u$  by  $-u$ ), so that

$$u(x_0) \geq u(x_2) \geq \dots \geq u(x_{2k}) \geq u(x_{2k+1}) \geq \dots \geq u(x_3) \geq u(x_1).$$

a) The first point  $y_1$  will now be replaced by one or two extremal points to get an upper bound of

$$\Phi(u(y_2) - u(y_1))$$

and then to increase the  $\Phi$ -variation. If  $y_1$  is already an extremal point, then we can skip this step and go directly to step b). Else, let  $i$  be the integer such that:  $x_i < y_1 < x_{i+1}$ . There are two cases:  $y_2 \in I_i$  or  $y_2 \notin I_i$ .

If  $y_2 \in I_i$ , then

$$|u(y_2) - u(y_1)| \leq |u(y_2) - u(x_i)|,$$

so that

$$\Phi(u(y_2) - u(y_1)) \leq \Phi(u(y_2) - u(x_i)).$$

So  $y_1$  will be replaced in the subdivision by  $x_i$ .

If  $y_2 \notin I_i$ , then

$$|u(y_2) - u(y_1)| \leq |\delta_i|,$$

so that

$$\Phi(u(y_2) - u(y_1)) \leq \Phi(u(x_{i+1}) - u(x_i)) = \Phi(\delta_i).$$

So  $y_1$  will be replaced by  $x_{i+1}$  and the point  $x_i$  is added as the first point of the subdivision. The subdivision is now  $x_i < x_{i+1} < y_2$ . Let  $k \geq i + 1$  such that  $x_k < y_2 \leq x_{k+1}$ . If  $k \geq i + 2$  then  $x_{i+2}, \dots, x_k$  are added to the subdivision in order to get a greater  $\Phi$ -variation.

Notice that the new subdivision starts from now on with one or some extremal points.

b) We will now get an upper bound of

$$\Phi(u(y_2) - u(x_k)) + \Phi(u(y_3) - u(y_2))$$

by removing  $y_2$  from the subdivision. There are two cases:  $y_3 \in I_k$  or  $y_3 \notin I_k$ .

If  $y_3 \in I_k$ , then  $y_2$  is simply removed from the subdivision since

$$\Phi(u(y_2) - u(x_k)) + \Phi(u(y_3) - u(y_2)) \leq \Phi(u(y_3) - u(x_k)).$$

If  $y_3 \notin I_k$ , there are two cases again.

If  $u(y_3)$  is between  $u(x_k)$  and  $u(y_2)$ , then

$$\Phi(u(y_2) - u(x_k)) + \Phi(u(y_3) - u(y_2)) \leq \Phi(\delta_k) + \Phi(u(y_3) - u(x_{k+1}))$$

since

$$|u(y_2) - u(x_k)| \leq |\delta_k| \text{ and } |u(y_3) - u(y_2)| \leq |u(y_3) - u(x_{k+1})|.$$

Else,  $u(y_2)$  is between  $u(x_k)$  and  $u(y_3)$  and it follows from (3.1) that

$$\Phi(u(y_2) - u(x_k)) + \Phi(u(y_3) - u(y_2)) \leq \Phi(u(y_3) - u(x_k)) \leq \Phi(\delta_k),$$

and then

$$\Phi(u(y_2) - u(x_k)) + \Phi(u(y_3) - u(y_2)) \leq \Phi(\delta_k) + \Phi(u(y_3) - u(x_{k+1})).$$

In both cases  $y_2$  is replaced by  $x_{k+1}$  and the  $\Phi$ -variation increases.

c) For  $y_3$  the situation is similar to that of the point b), since we have to find an upper bound of

$$\Phi(u(y_3) - u(x_{k+1})) + \Phi(u(y_4) - u(y_3)).$$

Continuing the process, the initial subdivision becomes a sequence of consecutive extremal points  $x_i, x_{i+1}, \dots, x_p$  with a greater  $\Phi$ -variation, less or equal to  $\sum_k \Phi(\delta_k)$ ,

which concludes the proof.  $\square$

#### 4. EXPLICIT ONE-DIMENSIONAL SOLUTIONS

In this section explicit solutions with almost minimal regularity are proposed. The regularity is simply and precisely estimated in  $BV^s$ , which is enough to get the correspondent Sobolev regularity ([1, 2]).

The one-dimensional problem considered is:

$$\partial_t u + \partial_x f(u) = 0, \quad u(0, x) = u_0(x).$$

If the flux  $f$  is smooth ( $f \in C^\infty$ ), nonlinear ( $\forall u, \exists k > 1, f^{(k)}(u) \neq 0$ ) and strictly convex but possibly degenerate, then the regularity in  $BV^\alpha$  with only an  $L^\infty$  initial data is already known ([1]). More precisely, if the flux degeneracy is  $d_f = d$ , then  $\alpha = \frac{1}{d}$  and the entropy solutions becomes immediately more regular:  $u(t, \cdot) \in BV_{loc}^\alpha$  as a function of  $x$  for all  $t > 0$ .

The point is now to show examples with no more regularity. The regularity is first considered in  $BV^s$  where the norm can be exactly computed. Second, the Sobolev regularity is studied at the end of this section.



**Proposition 12.** *Suppose that the nonlinear flux  $f \in C^\infty(K, \mathbb{R})$  satisfies Definition 2 with a degeneracy  $\alpha$ . Then for all  $\varepsilon > 0$ , for all  $T > 0$  there exists an entropy solution  $u$  such that for all  $t \in ]0; T[$ :*

$$u(t, \cdot) \in BV_{loc}^\alpha(\mathbb{R}) \cap W_{loc}^{\alpha-\varepsilon, 1}(\mathbb{R}) \quad \text{and} \quad u(t, \cdot) \notin BV_{loc}^{\alpha+\varepsilon}(\mathbb{R}) \cup W_{loc}^{\alpha+\varepsilon, 1}(\mathbb{R}).$$

Two examples are presented: the first one is a continuous solution with a small fractional regularity ([2]), the second one corresponds to an accumulation of Riemann problems. The  $BV^s$  or  $BV_\Phi$  estimations are precisely done on the initial data. The point is to have a time  $T_1 > 0$  before the waves interactions so that the  $BV^s$  norm remains constant on  $[0, T_1[$ . A change of variables  $Tt = T_1 \tau$ ,  $Tx = T_1 \xi$  yields a similar solution with a life span before waves interactions equals to  $T$ . In other words, if  $T_1$  is the life span for the entropy solution with initial data  $u_0(x)$  then with the initial data  $u_0\left(\frac{T_1}{T}x\right)$  the life span is  $T$ .

Example 13 is a continuous example not related to convexity. However, for a smooth flux with at least a non-zero derivative the function is locally left or right convex or concave. Example 14 uses the right convexity with a non-negative solution.

**Example 13.** Continuous example:  $u \in C^0([0, T] \times \mathbb{R})$ .

The following example is built in [2] where the critical  $s$ -total variation is estimated. Only the behavior of the initial data is recalled. If the maximal point of degeneracy of the flux is  $\underline{u} = 0$  then the explicit initial data is:

$$u_0(x) = x^a \cos\left(\frac{\pi}{x^b}\right) \quad \text{where } a = \alpha + \frac{\alpha^2}{\varepsilon}, \quad b = \frac{\alpha}{\varepsilon}.$$

In some sense the worst behavior of  $u$  is obtained with very high oscillations compensated precisely by a very flat behavior of  $u_0$  near the singular point  $x = 0$ .

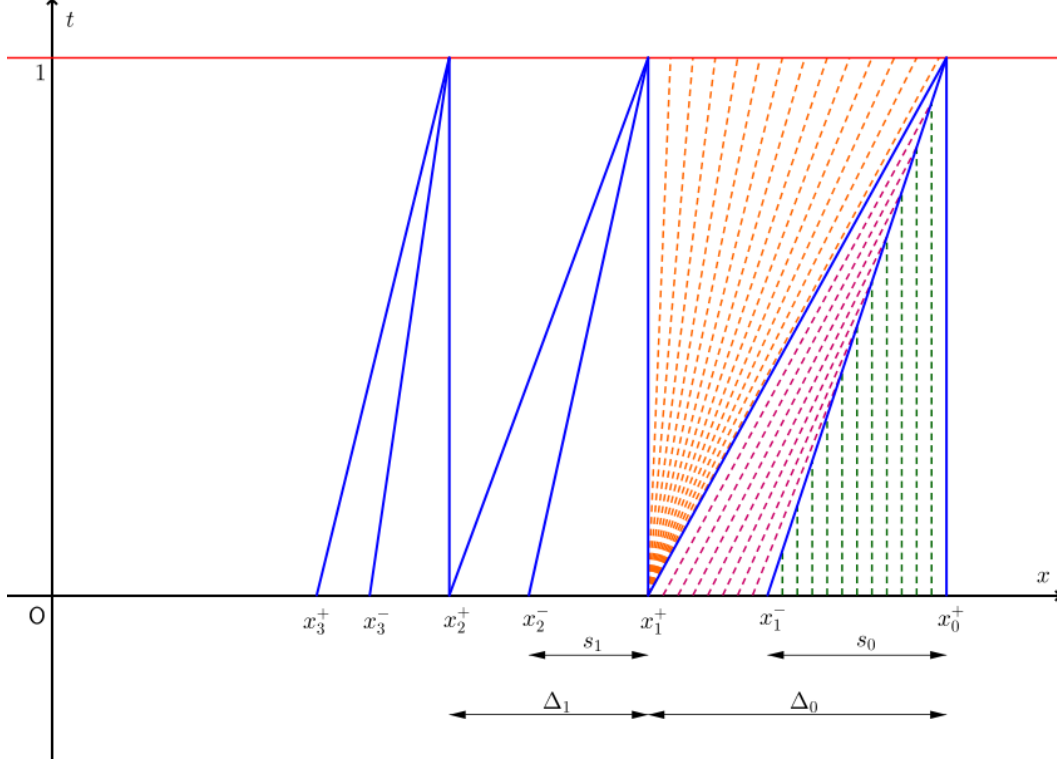
A classic way to build singular solutions is to take an initial piecewise constant data ([6] p. 13 and [11]). Thus the entropy solutions correspond to a succession of rarefaction waves and shock waves. The entropy solution is not continuous but we show that we can choose  $\varepsilon = 0$  in this context.  $BV^s$  appear to be the optimal spaces to study the regularity of entropy solutions [1, 3]. Since the study of the regularity of such solutions is not given in  $BV^s$  spaces but in Besov spaces in [11], a short study of the solution in  $BV^s$  is derived.

Proposition 12 can be improved in the following way: for all  $T > 0$ , there exists an entropy solution such that for all  $t \in ]0, T[$  and for all  $\varepsilon > 0$ ,  $u(t, \cdot) \in BV_{loc}^\alpha(\mathbb{R})$  and  $u(t, \cdot) \notin BV_{loc}^{\alpha+\varepsilon}(\mathbb{R})$ . In order to do so, we give Example 14. Such an improvement seems not so clear for the first continuous example.

**Example 14.** Piecewise elementary waves.

This example is presented in [11] (see also [6]) and studied in Besov spaces. A monotonic assumption is added to perform the  $BV_\Phi$  estimates thanks to Proposition 10. The construction in [11] is given for a power-law flux:  $f(u) = |u|^{1+d}$  for  $d \geq 1$ . This example is generalizable for any  $C^\infty$  flux satisfying Remark 5 with the same  $d$ ,  $f'(0) = 0$  and

$f$  is strictly convex in a right neighborhood of 0. To have  $f'(0) = 0$  it is enough to make a change of space variable:  $x \longleftrightarrow x - f'(0)t$ . For the right local convexity it is assumed that  $f^{(1+d)}(0) > 0$ . In the concave case  $f^{(1+d)}(0) < 0$  the example can be easily modified with the same picture and negative wave speeds. A suitable piecewise constant initial data is defined.



Since  $f$  is convex on  $[0, +\infty[$ , only decreasing jumps satisfy the Lax' entropy condition. Increasing jumps will be replaced by rarefaction waves. Let  $(c_k)_{k \geq 1} \in l^d$  be a sequence of positive numbers. Set for  $k \in \mathbb{N}$ :

$$\Delta_k := a(c_{k+1}), \quad s_k = \frac{f(c_{k+1})}{c_{k+1}} < \Delta_k$$

$$x_k^+ := \sum_{j=k}^{+\infty} \Delta_j < +\infty, \quad x_{k+1}^- := x_k^+ - s_k > x_{k+1}^+$$

$$u(\cdot, 0) := \sum_{k=1}^{+\infty} c_k \chi_{I_k} \quad \text{where} \quad I_k := [x_k^+, x_k^-].$$

As explained in [11], an initial jump connecting 0 to  $c_k$  evolves into a rarefaction wave whose leading edge moves with speed  $a(c_k) \sim \lambda c_k^d$ , where  $\lambda = \frac{f^{(1+d)}(0)}{d!} > 0$ . The choice of  $\Delta_k$  made here ensures that all waves do not interact in the time interval  $]0; T[$  with  $T = 1$ .

The interaction times can be calculated explicitly: the left rarefaction and the right shock intersect at time  $t_k^- = \frac{x_{k+1}^- - x_{k+1}^+}{a(c_{k+1}) - s_k} = 1$ ; the left shock and the right rarefaction intersect at time  $t_k^+ = \frac{x_k^+ - x_{k+1}^-}{s_k} = 1$ .

We now focus on two different choices of the sequence  $(c_k)$  and prove below the stated results:

- (1) If  $c_k = \frac{1}{k^{\alpha+\varepsilon}}$  for all  $k > 0$ , then  $u(t, \cdot) \in BV^s$  for all  $s < \alpha + \varepsilon$  but  $u(t, \cdot) \notin BV^{\alpha+\varepsilon}$ .
- (2) If  $c_k = \frac{1}{(k \ln^{1+\eta}(k))^\alpha}$  for all  $k > 1$ ,  $\eta > 0$ , then  $u(t, \cdot) \in BV^\alpha$  but  $u(t, \cdot) \notin BV^s$  for all  $s > \alpha$ .

These examples present oscillations with decreasing amplitudes, so the regularity is computed simply in the spaces  $BV^s$  or  $BV_\Phi$  thanks to Proposition 10. Notice that:

- $TV_\Phi u = 2 \sum_k \Phi(c_k)$  so that  $u$  belongs to  $BV_\Phi$  if and only if  $\sum_k \Phi(c_k) < \infty$ .
- $BV^s$  is simply the  $BV_\Phi$  space with the function  $\Phi(y) = y^{1/s}$  for  $y \geq 0$ .
- before the interaction time of waves  $T$ , the  $BV^s$  norm of the entropy solution is equal to that of the initial data.

- (1) The first example is related to the convergence of the series  $\sum_k \left( \frac{1}{k^{\alpha+\varepsilon}} \right)^{1/s}$  for  $s < \alpha + \varepsilon$  and the divergence of the harmonic series.
- (2) The second example works with the same arguments. Note that the regularity can be estimated more precisely in the  $BV_\Phi$  spaces. If  $\Phi(y) = y^d |\ln^\gamma y|$  for  $y > 0$ , with  $d = \frac{1}{\alpha}$ , then:

$$\Phi(c_k) = \frac{\alpha^\gamma |\ln(k) + (1 + \eta) \ln(\ln(k))|^\gamma}{k \ln^{1+\eta}(k)} \sim \frac{\alpha^\gamma}{k \ln^{1+\eta-\gamma}(k)},$$

so that the series  $\sum_k \Phi(c_k)$  converges if and only if  $\gamma < \eta$ . Since the following strict inclusions hold for all  $\varepsilon > 0$ ,  $\gamma > 0$ :  $BV^{s+\varepsilon} \subsetneq BV_{y^{1/s} |\ln^\gamma y|} \subsetneq BV^s$  ([3]), it follows that  $u(t, \cdot) \in BV_{y^{1/\alpha} |\ln^\gamma y|} \subset BV_{y^{1/\alpha}} = BV^\alpha$  for all  $0 < \gamma < \eta$  and  $u(t, \cdot) \notin BV_{y^{1/\alpha} |\ln^\eta y|}$ . These estimates are valid for all  $0 \leq t < T$ .

We now turn to Sobolev estimates. For all  $\varepsilon > 0$ ,  $BV_{loc}^s \subset W_{loc}^{s-\varepsilon, 1}$  ([1]), so the last point is to show that the previous examples do not belong to  $W_{loc}^{\alpha+\varepsilon, 1}$ .

**Lemma 15.** *Let  $u$  a piecewise constant function on  $]0, 1[$ ,  $(x_k)$  a decreasing sequence such that  $x_0 = 1$  and  $x_k \rightarrow 0$ . Set for  $k \geq 1$   $I_k = ]x_k, x_{k-1}[$ ,  $\Delta_k = x_{k-1} - x_k$ ,  $u(x) = u_k$  on  $I_k$  and  $c_k = |u_{k-1} - u_k|$ . Assume that  $(\Delta_k)$  is a decreasing sequence. Let  $s \in ]0, 1[$ , if  $\sum_k c_k \Delta_k^{1-s} = +\infty$  then  $u \notin W^{s, 1}(]0, 1[)$ .*

*Proof.* It suffices to roughly estimate the  $W^{s,1}$  semi-norm of  $u$ :

$$\begin{aligned}
|u|_{W^{s,1}([0,1])} &= \int_0^1 \int_0^1 \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dx dy \\
&= \sum_{k \geq 1} \int_{x_k}^{x_{k-1}} \int_0^1 \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dy dx \\
&\geq \sum_{k \geq 2} \int_{x_k}^{x_{k-1}} \int_x^{x+\Delta_k} \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dy dx \\
&= \sum_{k \geq 2} \int_{x_k}^{x_{k-1}} \int_{x_{k-1}}^{x+\Delta_k} \frac{|u(x) - u(y)|}{|x - y|^{1+s}} dy dx \\
&\geq \sum_{k \geq 2} \int_{x_k}^{x_{k-1}} \frac{c_k}{\Delta_k^{1+s}} (x - x_k) dx \\
&= \frac{1}{2} \sum_{k \geq 2} c_k \Delta_k^{1-s}.
\end{aligned}$$

□

Example 14 is not piecewise constant but, for any  $t > 0$  fixed, it suffices to consider  $\underline{u}(t, x)$  the function which is equal to  $u(t, x)$  when  $u(t, x)$  is locally zero and on the rarefaction with maximal value  $c_k$ ,  $\underline{u}(t, x) = \frac{1}{2}c_k$  if  $u(t, x) > \frac{1}{2}c_k$  and  $\underline{u}(t, x) = 0$  else. Since the  $W^{s,1}$  semi-norm of  $\underline{u}$  is less than the one of  $u$  the conclusion of Lemma 15 holds. The same method can also be used for continuous Example 13. More precisely, let  $x_k$  be the decreasing sequence of roots of  $u$  and  $c_k$  the supremum of  $|u|$  on  $I_k$ . Set  $\underline{u}(t, x) = \frac{1}{2}c_k$  if  $u(t, x) > \frac{1}{2}c_k$  and 0 else on  $I_k$ .

We can now achieve the proof of Proposition 12. For the Riemann series of Example 14 with  $s = \alpha + \varepsilon$ , we can write:  $\sum_k c_k \Delta_k^{1-s} \sim \lambda \sum_k c_k^{1+d(1-s)} \rightarrow +\infty$  since  $1+d(1-\alpha) = d$ . Thus, applying an extension of Lemma 15, it follows that  $u(t, \cdot) \notin W^{s+\varepsilon,1}([0,1])$ . The other examples conclude the proof similarly.

We do not go further in the Sobolev framework since the best  $W^{s,p}$  was already conjectured in [11] with  $p = \frac{1}{s}$  and was reached in [1] for some convex fluxes. We now turn to the multidimensional case.

## 5. MONOPHASE ENTROPY SOLUTION

An idea to build the most singular solution follows the geometric optics study. Such a method provides a family of solutions depending on very high frequencies. In this framework the singularity of the whole family (uniform Sobolev bounds) is given by the relation between the small amplitude and the wavelength [16]. It is known that the most singular case occurs near some constant state [4, 14, 16, 17]. Moreover, in [4] for the worst case, the multi-phase expansion near the constant state has only one phase with the highest frequency. A monophasic expansion is exploited in [16, 17] to get the supercritical geometric optics expansions. This remark is also a key point to

build individual solution (and not a whole family) with the almost minimal regularity expected. It is then expected that a solution with one phase  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  carefully chosen can yield a solution with low regularity:

$$U(t, X) = \underline{U} + u(t, \varphi(X)).$$

$\underline{U}$  is the point where the vectorial flux  $F$  is locally the less nonlinear (see Definition 4). The function  $u(t, x)$  solves a one-dimensional conservation law where the scalar flux  $f$  is:

$$f(u) = \nabla \varphi \cdot F(\underline{U} + u).$$

The computation of the flux is a direct application of the chain rule formula for smooth solutions. If the solution is not smooth this formula is still valid for weak entropy solutions [4, 15]. The following classic lemma is stated without proof.

**Lemma 16.** *If  $u$  is an entropy solution of  $\partial_t u + \partial_x f(u) = 0$  with the scalar flux  $f(u) = \nabla \varphi \cdot F(\underline{U} + u)$  then  $U(t, X) = \underline{u} + u(t, \varphi(X))$  is an entropy solution of  $\partial_t U + \operatorname{div}_X F(U) = 0$ .*

Let us choose the critical phase  $\varphi$  taking account of the derivatives of  $f$ :

$$f^{(k)}(0) = \nabla \varphi \cdot F^{(k)}(\underline{U}).$$

The phase is chosen to have the most degenerate scalar flux  $f$ . Since by Definition 4  $\operatorname{rank} \{F''(\underline{U}), \dots, F^{(d_F)}(\underline{U}), F^{(1+d_F)}(\underline{U})\} = n$  and  $\operatorname{rank} \{F''(\underline{U}), \dots, F^{(d_F)}(\underline{U})\} = n - 1$ , there is only one direction to choose  $\nabla \varphi$  such that the scalar flux  $f$  has the same degeneracy than the vectorial flux  $F$ :  $d_f = d_F$ . This only way is to take  $\varphi$  such that:

$$0 \neq \nabla \varphi \perp \{F''(\underline{U}), \dots, F^{(d_F-1)}(\underline{U})\}.$$

Thus, up to a normalization, the choice of the linear phase  $\varphi$  is unique. Now it suffices to take an initial data  $u_0$  with a low regularity from the one-dimensional case with  $u_0$  small enough to get a critical solution with the critical initial data:

$$U(0, X) = U_0(X) = \underline{U} + u_0(\varphi(X)).$$

The entropy solution  $u$  is chosen to have the expected low regularity in  $BV^s$  and  $W^{s,1}$  thanks to Proposition 12. In Sobolev spaces the same low regularity is inherited by  $U$  on the same interval of time. Precisely, a linear change of variables  $X \mapsto Y = (Y_1, \dots, Y_n)$  in  $\mathbb{R}^n$  does not change the best Sobolev exponent. The change of variables is chosen to have  $Y_1 = \varphi(X)$ . The optimal Sobolev regularity has to be estimated on function depending only on  $Y_1$ :

$$U(t, Y) = \underline{U} + u(t, Y_1) = v(t, Y_1).$$

The low regularity of the entropy solution  $U$  and then Theorem 1 follow from this classical lemma:

**Lemma 17.** *If  $U(Y) = v(Y_1)$ ,  $\varepsilon > 0$ ,  $v \in W_{loc}^{s-\varepsilon,1}(\mathbb{R}, \mathbb{R})$  but  $v \notin W_{loc}^{s+\varepsilon,1}(\mathbb{R}, \mathbb{R})$ , then  $U \in W_{loc}^{s-\varepsilon,1}(\mathbb{R}^n, \mathbb{R})$  but  $U \notin W_{loc}^{s+\varepsilon,1}(\mathbb{R}^n, \mathbb{R})$ .*

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